Existence of a long time scale in quantum chaos

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An analysis of a popular model, the quantum kicked top, is presented based on the entropy of a quantum state. It is shown that typically the quantum chaos persists on a long time scale $t_R \propto \hbar^{-2}$. A simple estimate for the crossover from a short (logarithmic) to a long (diffusion) time scale is derived. [S1063-651X(97)06406-4]

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In a recent paper [1] Alicki, Makowiec, and Miklaszewski presented one more confirmation for the existence of the short (logarithmic) time scale in quantum chaos using a simple model of the kicked quantum top and a finite-time analog of the classical dynamical Kolmogorov-Sinai (KS) entropy. This random time scale $t_r \sim \ln \hbar^{-1}$, on which the quantum motion is similar to the classical one including the exponential instability, had been discovered in Ref. [2] and was subsequently confirmed and studied further in many papers [3–5] (see also Ref. [6]).

The main purpose of this paper is to point out that in addition to this short time scale there is generally another one t_R, which is much longer ($\ln t_R \sim \ln \hbar^{-1}$) and on which a partial quantum-classical correspondence persists, namely, the quantum diffusion closely follows the classical one even though the former is dynamically stable [7] (see also Refs. [3,5]). The absence of the long scale t_R in the model in [1] is a result of a special choice for one model's parameters value.

Generally, the quantum top is described [8] by the unitary operator (per kick, $\hbar = 1$)

$$U(p,k,j) = e^{-ikJ_z^2/2j} e^{-ipJ_y}, \tag{1}$$

which depends on two classical parameters k and p and one quantum parameter $j \gg 1$ (in quasiclassics). In Ref. [1] the value $p = \pi/2$ was chosen following Ref. [8], in which the only reason for such a choice was merely to simplify the quantum map. This particular choice leads to a nongeneric, fast (''ballistic'') relaxation to the ergodic steady state. According to data in Fig. 2 [1], the relaxation time $t_{er}(p) \approx 1.5$ iterations only, in this case. Moreover, some relaxation occurs even for almost regular motion ($k = 1$; see Fig. 1 in Ref. [1]).

On the contrary, if $p \ll 1$ the relaxation becomes diffusive and relatively slow, and only for chaotic motion, of course, namely, when the parameter $K = pk > 1$. In the simplest case $|J_z| \ll j$ the diffusion rate in J_z is [8,9]

$$D \approx \frac{1}{2}(pj)^2 C(K) \sim (pj)^2, \tag{2}$$

where $C(K) \sim 1$ accounts for dynamical correlations. Hence the relaxation time (in number of kicks) is $t_{er}(p) \sim j^2/D \sim p^{-2} \gg 1$. During the relaxation process the quantum entropy keeps growing until it reaches the maximal value H_{er} for the ergodic state:

$$H(t) \equiv -\sum f(J_z,t) \ln f(J_z,t) \to H_{er} = \ln(2j+1), \tag{3}$$

where J_z are integers and $f(J_z) = |\psi(J_z,t)|^2$ is the distribution function.

For an initially narrow Gaussian distribution the entropy in the diffusion regime is $H_D(t) \approx \ln(2\pi eDt)/2$, assuming $J_z \gg 1$ and $D(J_z) \approx$ const [9]. This entropy growth is much slower than that on the random time scale $t < t_r$ [$H_r(t) = th_r$] and the corresponding KS entropy vanishes [10], as it should for a quantum motion with a discrete spectrum.

Notice that in the classical limit the entropy would grow indefinitely with constant rate $h_r = \Lambda \approx \ln(K/2)$ due to the continuity of the variable J_z as explained in the beginning of Ref. [1] (see also Ref. [11]). In the quantum case the classical instability h_r is restricted to the short time scale t_r, which can be found approximately from the equation $th_r(t_r) = H_D(t_r)$. This gives an asymptotic ($j \to \infty$) estimate $t_r \approx \ln(pj)/\ln(pk)$, in agreement with previous results [2–5].

The final steady state is ergodic with entropy (3) only under the additional condition [3,5,9] $t_{er} \ll t_H = (2j+1)/2\pi = \exp(H_{er})/2\pi$ or $jp^2 \gg 1$, where t_H is the mean quasienergy level density, also called the Heisenberg time. In the opposite case ($jp^2 \lesssim 1$) the quantum diffusion is restricted to the relaxation (diffusion) time scale [3,5,9]

$$t_R \sim D \sim (pj)^2 \lesssim t_H. \tag{4}$$

Hence, the quantum steady state is essentially nonergodic due to localization of quantum diffusion. Assuming approximately exponential localization with a characteristic length $l \approx D$, the final steady-state entropy in this case is $H_l \approx 1 + \ln D \to 2\ln(pj) \lesssim H_{er}$.

The diffusive time scale t_R (4), which is our main interest here, is always much longer than the instability scale t_r. Only for $t \gg t_R$ is the motion completely dominated by the quantum effects.

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